Comments on "Natural Frequencies and Dampings Identification using Wavelet Transform: Application to Real Data"

(A letter to the editors of the Journal of Mechanical Systems and Signal Processing)

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1 INTRODUCTION

Authors M. Ruzzene *et alii* [[Ruzzene et al.(1997)]] present in their article a very useful bridge from the Hilbert Transform method for vibration analysis to the Wavelet Transform. The conceptual relationship between the two is made in a refreshingly lucid manner; nonetheless some small technical *errata* exist, obfuscating the theoretical development as originally presented. This letter attempts to address some of these issues in order to clear up any potential confusion for the benefit of other close readers of this important work.

The independent article by H. Dishan [[Dishan(1996)]] is also a useful guide bridging the Hilbert and Wavelet transforms. Together with the article by Ruzzene, a formidable signal processing method is presented. We seek to conjoin these two works as well as clarify the finer points of the original manuscripts.

The notation used in this letter is consistent with the original, aside from some crucial departures which are duly noted. Equations from the original article are prefixed by either of the letters "R" or "D", respectively, after the namesakes of the original authors, Ruzzene *et alii* and/or Dishan.

1.1 The Morlet Wavelet

The Morlet Wavelet is used as the basis function for the considered wavelet transformation. The form adopted by the original authors is that given by Ruzzene equation (R2). This equation for Morlet's Wavelet is a specific instance of the more general function

$$g(t) \triangleq C \exp\left\{-\left(\alpha t^2 + \beta t + \gamma\right)\right\} \ , \tag{1}$$

where C, α, β, and γ are some constants. When this function is first translated (time-shifted) by some amount b, and then dilated (scaled) by some amount a, we can write

$$\begin{aligned} g\left(\frac{t-b}{a}\right) &= C \exp\left\{-\left[\alpha\left(\frac{t-b}{a}\right)^2 + \beta\left(\frac{t-b}{a}\right) + \gamma\right]\right\} \\ &= C \exp\left\{-\left[\alpha\left(\frac{t}{a}\right)^2 + \left(\beta - \frac{2\alpha b}{a}\right)\left(\frac{t}{a}\right) + \left(\alpha\frac{b^2}{a^2} - \beta\frac{b}{a} + \gamma\right)\right]\right\} . \end{aligned} \tag{2}$$

Selection of the values $a = 1$ and $b = 0$ suppress the dilation and translation properties of the wavelet, respectively, yielding the so-called *mother wavelet*. Both Ruzzene *et alii* [Ruzzene et al.(1997), eq. R2] and Dishan [Dishan(1996), eqs. D8, D10] use the Morlet Wavelet mother obtained by first substituting the parameter values

$$C = 1/\sqrt{a} \ , \qquad \alpha = 1/2 \ , \qquad \beta = -j\omega_0 \ , \qquad \gamma = 0 \tag{3}$$

into equation (2), yielding

$$g\left(\frac{t-b}{a}\right) = \frac{1}{\sqrt{a}} \exp - \left[\left\{\frac{1}{2}\left(\frac{t}{a}\right)^2 - \left(j\omega_0 + \frac{b}{a}\right)\frac{t}{a} + \left(j\omega_0 + \frac{b}{2a}\right)\frac{b}{a}\right]\right\} , \tag{4}$$

and then setting $a = 1$ and $b = 0$ to ultimately produce

$$g(t) = e^{j\omega_0 t} e^{-t^2/2} \ , \tag{5}$$

where ω_0 is the *centre frequency* of the wavelet.[1] Also note that the value of C used by Ruzzene (as defined in (3)) differs from that used by Dishan to obtain equation (D11).

1.2 The Morlet Wavelet Transformation

The wavelet transformation under consideration is proportional to the convolution of this *analysing wavelet* $g((t-b)/a)$ with some signal $x(t)$.[2] As demonstrated in equation (R3), this convolution may be performed either directly or, alternatively, via the convolution theorem of the Fourier Transform, which defines the (inner product) transform pair (*exempli gratia* [Press et al.(1992), eq. 12.0.9])

$$\mathrm{Conv}(x, g) \ = \ <x(t), g(t)> = \ x(t) \star g(t) \quad \Longleftrightarrow \quad X(\omega) \cdot G(\omega) \ . \tag{6}$$

Furthermore, the correlation theorem of the Fourier Transform defines the transform pair (*exempli gratia* [Press et al.(1992), eq. 12.0.10]) [3]

$$\mathrm{Corr}(x, g) \quad \Longleftrightarrow \quad X(\omega) \cdot G(-\omega) \ = \ X(\omega) \cdot G^*(\omega) \ , \tag{7}$$

which estimates the similarity of the frequency content between $X(\omega)$ and $G(\omega)$. Since the wavelet has a Gaussian window (embodied by its exponential envelope), the frequency

[1]—Not to be confused with the customary denotation for the natural frequency of the vibration being analysed in $x(t)$.
[2]Note that Dishan uses the variables h and f in place of g and x.
[3]The equality on the right-hand side is possible because x is real-valued and g is composed of an even real component and an odd imaginary component.[Press et al.(1992), §12.0]

correlation is also localised in time. This dual time-frequency localisation is the essential usefulness provided by signal processing using the wavelet transformation.

The wavelet transformation can thus be written in terms of the inverse Fourier Transform of the spectra $X(\omega)$ and $G^*(\omega)$: [4]

$$W \triangleq \mathcal{F}^{-1}\left\{\mathcal{F}\left\{x(t)\right\} \cdot \mathcal{F}^*\left\{g(t)\right\}\right\} = \mathcal{F}^{-1}\left\{X(\omega)G^*(\omega)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} X(\omega)\cdot G^*(\omega)e^{j\omega t}\,\mathrm{d}\omega\ , \tag{8}$$

where G^* is the complex conjugate of the Fourier Transform

$$G(\omega) \triangleq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g\left(\frac{t-b}{a}\right) e^{-j\omega t}\,\mathrm{d}t\ . \tag{9}$$

Using the Gaussian Fourier pair [Hahn(1996), §1.7.1], [Spiegel(1968), #15.75]

$$\int_{-\infty}^{+\infty} C\exp\left\{-\left(px^2+qx+r\right)\right\}\mathrm{d}x = C\sqrt{\frac{\pi}{p}}\exp\left\{\frac{q^2-4pr}{4p}\right\} \tag{10}$$

and substituting equation (2) into (9), it can be shown that

$$G(\omega) = \frac{aC}{\sqrt{2\alpha}}\exp\left\{\frac{(\beta+j\omega a)^2}{4\alpha}\right\}\exp\left\{-(\gamma+j\omega b)\right\}\ . \tag{11}$$

The conjugate of the Fourier Transform of equation (4) is now

$$G^*(\omega) \triangleq \sqrt{a}\exp\left\{-\frac{1}{2}(a\omega-\omega_0)^2\right\}e^{+j\omega b}\ . \tag{12}$$

In tandem, equation (8) becomes

$$W = \sqrt{\frac{a}{2\pi}}\int_{-\infty}^{+\infty} X(\omega)\exp\left\{-\frac{1}{2}\left(a\omega-\omega_0\right)^2\right\}\exp\left\{j\omega\left(t+b\right)\right\}\mathrm{d}\omega\ , \tag{13}$$

where the signal spectrum $X(\omega)$ is typically computed via the fast Fourier Transform algorithm.

This result differs from equation (R3), which neglects the exponential term $e^{j\omega t}$ inherent to the inverse Fourier Transform, the scaling factor $\sqrt{a}$, and also the factor of $\sqrt{2\pi}$ from one or the other of the transform $G(\omega)$ or W (these factors are, however, evident in Dishan's equation (D11)). Moreover, all references to $G(\omega)$ in the original work by Ruzzene *et alii* should have the relevant exponential *halved* as above, in accordance with other references on the Morlet wavelet.

2 WAVELET TRANSFORMATION OF AN ANALYTIC SIGNAL

When the mathematical form of the signal $x(t)$ under analysis is known (or assumed), a closed analytical form of the wavelet transformation may be derived by evaluating the (correlation)

[4] We use the *symmetric* form of the Fourier Transform definition, which premultiplies both the *forward and inverse* transformations by a factor of $1/\sqrt{2\pi}$, *exempli gratia* [Strang(1986), eq. 4.3.12].

convolution integral directly [Ruzzene et al.(1997), eq. R1] [Dishan(1996), eq. D1]:

$$W = \int_{-\infty}^{+\infty} x(t)\, g\left(\frac{t-b}{a}\right)\, \mathrm{d}t \ . \tag{14}$$

However, generally the Fourier Transform of the conjugate is not necessarily equivalent to the complex conjugate of the Fourier Transform, so in equation (R1) the conjugate g^* is premature and should be postponed until afterwards, when it is applied to $G(\omega)$, giving $G^*(\omega)$. Dishan observes this principle in his equation (D1).

Ruzzene *et alii* aptly present the notion of an *analytic signal* (eq. (R17)) [Hahn(1996), §1.13]

$$z(t) = k(t)e^{j\phi(t)} \ , \tag{15}$$

where the *instantaneous amplitude*

$$k(t) \triangleq A \exp \int_0^t -\sigma(t)\, \mathrm{d}t \tag{16}$$

and the *instantaneous phase*

$$\phi(t) \triangleq \int_0^t \omega(t)\, \mathrm{d}t + \phi_0 \ . \tag{17}$$

The analytic signal (15) may now be written

$$z(t) = Ae^{j\phi_0} \exp \int_0^t \{-\sigma(t) + j\omega(t)\}\, \mathrm{d}t = Ae^{j\phi_0} \exp \int_0^t s(t)\, \mathrm{d}t \ , \tag{18}$$

where the *angular complex frequency* [5]

$$s(t) \triangleq -\sigma(t) + j\omega(t) \ . \tag{19}$$

There is some confusion in the nomenclature of the original article with regard to the instantaneous phase $\phi(t)$, and its derivative, the instantaneous angular frequency. The equations (R13) - (R15), (R20), (R21), and (R24) refer to the *angular frequency* $\omega(t) = \dot{\phi}(t)$, *not* its integral, the instantaneous phase, $\phi(t)$ as is written.

2.1 Application to Second-Order Systems

The definitions for instantaneous radial velocity (more commonly called the *rate of (amplitudinal) decay*)

$$\sigma(t) = \zeta\omega_n \tag{20}$$

and instantaneous phase

$$\phi(t) = \omega_d t + \phi_0 \tag{21}$$

given by Ruzzene *et alii* in equations (R17) and (R19), respectively, are the appropriate substitutions for the time-invariant, second-order oscillation produced by

$$z(t) = Ae^{j\phi_0} e^{\int_0^t s(t)\, \mathrm{d}t} = Ae^{-\zeta\omega_n t} e^{j(\omega_d t + \phi_0)} \ , \tag{22}$$

[5]—Also known as the Laplace operator s.

where ω_n is the *undamped natural frequency* (or *mode*) of the system, related to the *damped natural frequency*

$$\omega_d \triangleq \omega_n \sqrt{1-\zeta^2} \tag{23}$$

by the *viscous damping coefficient* ζ.

Dishan shows that the wavelet transformation can be used to construct a passband of Hilbert Transformers across the frequency range of interest. Ruzzene *et alii* show that, alternatively, the wavelet transformation can be used to "focus in" on frequencies of interest, by tuning the wavelet centre frequency ω_0. The necessary stipulation is that the dilation

$$a \longrightarrow \omega_0/\omega(t) = \omega_0/\dot{\phi}(t) \; . \tag{24}$$

In practise, a first iteration of equations (R24) with some dilation a will yield a fair estimate of ϕ for the frequency with the highest correlation for that initial value of a, which can be used in (24) above to iterate a. Usually two iterations are found to be sufficient to focus the dilation almost precisely onto the nearest frequency, validating equation (R28). Notice that in this case the exponential term containing the dilation parameter a approaches unity, explaining why the algorithm works as published even though the exponential should be halved as explained before.

The phase defined in equation (R15) conflicts with that in equation (R23). Remember that the values intended are actually angular frequencies $\omega = \dot{\phi}$, not the instantaneous phase ϕ as is shown. This matter is cleared up with the observation that the wavelet transformation shown in equations (R11) and in (R14) should read

$$W = \sqrt{a}k(t) \exp\left\{-\frac{1}{2}\left(a\omega_d - \omega_0\right)^2\right\} \exp\left\{j\omega_d(t+b)\right\} = \sqrt{a}k(t) \exp\left\{-\frac{1}{2}\left(a\dot{\phi} - \omega_0\right)^2\right\} \exp\left\{j\dot{\phi}(t+b)\right\} \tag{25}$$

Now the dilation tuning described by equation (24) is evident, as well as the phase relation of equation (R15), which should read

$$\angle W = \dot{\phi}(t)\,(t+b) = \omega_d(t+b) \; , \tag{26}$$

which is simply equal to $\omega_d t$ when the time shift $b = 0$ as is the case for the examples shown in the original. Thus the time derivative of the phase angle of the wavelet yields the damped natural frequency ω_d of the mode focused by dilation parameter a. The magnitude of this mode is then simply

$$k(t) = \frac{1}{\sqrt{a}}\,|W| = \sqrt{\frac{\dot{\phi}}{\omega_0}}\,|W| = \sqrt{\frac{\omega_d}{\omega_0}}\,|W| = \frac{1}{\sqrt{\omega_0}}\,|W|\,\frac{\mathrm{d}}{\mathrm{d}t}\angle W \; . \tag{27}$$

The negative slope of the natural logarithm of amplitude $k(t)$ is then equal to the decay rate σ. Once the decay rate and damped natural frequency are known, the damping ratio ζ and undamped modal frequency ω_n may be determined via the relations given in (20) and (23).

3 CONCLUSION

The work of Ruzzene *et alii*, together with that of Dishan, bridge the relationship between the Hilbert Transform and Morlet Wavelet Transformation in a manner most useful for the modal analysis of linearly separable, multimodal oscillatory systems. Nonlinear viscous damping and frequency, or distributed linear frequencies and associated modal damping, are both made possible, in a computationally fast manner using the Hilbert Transform, or alternatively in a clean, noiseless manner of signal processing by the wavelet transformation. Some minor discrepancies within the original papers are treated to establish consistency between them.

The authors hope the reader will find this review of the details useful for application of these powerful methods to their actual data. Interested readers are encouraged to send e-mail with any comments or questions.

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