Asymmetric Viscous and Kinetic Friction Identification

via the Extended Logarithmic Decrement Method

(An article for submission to the ASME Journal of Dynamic Systems, Measurement, and Control)

Julian A. de Marchi and Kevin C. Craig
Department of Mechanical Engineering, Aeronautical Engineering & Mechanics
Rensselær Polytechnic Institute
Troy, New York 12180-3590
mechatronics@rpi.edu   www.meche.rpi.edu/research/mechatronics

ABSTRACT

We extend the traditional time-domain identification technique known as the *logarithmic decrement* method to include estimation of asymmetric kinetic and viscous friction. The method may be applied to the free vibration response of a sufficiently underdamped, linear, second-order system with time-invariant physical parameters; only the time history of displacement data is needed for the identification. The technique is demonstrated in theory, and verified by simulation.

INTRODUCTION

Simple Harmonic Motion

Consider the free (homogeneous) harmonic oscillation of the piecewise linear, time-invariant, second-order system

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + f_c = F_0 \ , \tag{1}$$

where in order to produce an underdamped system oscillation, customarily a step input force F_0 is the system input.

Coulomb friction [Coulomb (1785)] is a piecewise constant function defined in terms of static and kinetic friction:

$$f_c = \begin{cases} f_s & \dot{x}(t) = 0 \\ f_k \operatorname{sgn}(\dot{x}) & \dot{x}(t) \neq 0 \end{cases} \ , \tag{2}$$

where due to the nature of friction $f_s \geq f_k$ (ergo, the counter-intuitive phenomenon of stick-slip friction, or *stiction*).

Substituting $\ddot{x}(t) = \dot{x}(t) = 0$ into (1), the static friction satisfies

$$x(t) \leq x_s \quad \text{where} \quad x_s \triangleq f_s/k \ . \tag{3}$$

This describes a region of possible positions x for which the spring force kx is insufficient to produce movement by overcoming the static friction f_s. The static friction is important, but let us now first consider the kinetic friction on the system.

Notice that equation (1) is linear when either $\dot{x} < 0$ or $\dot{x} > 0$. Defining

$$x_k \triangleq f_k/k \qquad \text{and} \qquad x_0 \triangleq F_0/k \ , \tag{4}$$

we can thus reformulate the problem in terms of the piecewise linear equation

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = 0 \ , \tag{5}$$

where

$$y(t) \triangleq x(t) + x_k \operatorname{sgn} \dot{x}(t) - x_0 \qquad \text{and} \qquad \dot{y}(t) \equiv \dot{x}(t) \;, \tag{6}$$

within regions of unidirectional motion (those for which $\dot{x}$ does not change sign).

Underdamped Response

The underdamped, transient response may be expressed as

$$y(t) = Ae^{-\sigma t} \sin(\omega_d t - \phi_0) \tag{7}$$

where the *rate of decay*

$$\sigma \triangleq \frac{c}{2m} \;, \tag{8}$$

which relates the *underdamped natural frequency*

$$\omega_d \triangleq \sqrt{\omega_0^2 - \sigma^2} \;, \tag{9}$$

where $\sigma^2 < \omega_0^2$, to the *undamped natural frequency*

$$\omega_0 \triangleq \sqrt{\frac{k}{m}} \;. \tag{10}$$

The oscillation has an *amplitude* (also called *envelope*)

$$A^2 = \frac{\dot{y}_0^2 + 2\sigma \dot{y}_0 y_0 + \omega_0^2 y_0^2}{\omega_d^2} \tag{11}$$

and *phase*

$$\tan \phi_0 = -\frac{\omega_d y_0}{\dot{y}_0 + \sigma y_0} \;, \tag{12}$$

given the initial conditions on position and velocity

$$y_0 \triangleq y(t_0) \qquad \text{and} \qquad \dot{y}_0 \triangleq \dot{y}(t_0) \;. \tag{13}$$

These equations are valid for all underdamped, second-order oscillations, provided that the physical quantities $m > 0$, $c \geq 0$ and $k > 0$.

Asymmetric Harmonic Motion

Many real systems exhibit appreciable asymmetric Coulomb and viscous friction, which can be defined as

$$f_s = \bar{f}_s + \Delta f_s \operatorname{sgn} F_r(t) \tag{14a}$$

$$f_k = \bar{f}_k + \Delta f_k \operatorname{sgn} \dot{x}(t) \tag{14b}$$

$$c = \bar{c} + \Delta c \operatorname{sgn} \dot{x}(t) \tag{14c}$$

where the terms with a $\overline{bar}$ denote the mean frictional values, and those with the Δ denote their variation depending on the direction of the residual force $F_r(t)$ or motion $\dot{x}(t)$.

The asymmetric Coulomb friction can be expressed in terms of the parameters x_s and x_k as before, in which case the asymmetries give rise to a constant offset in the displacement response x(t) as per (6). The system's displacement may thus be written in the more general form

$$\begin{aligned} y(t) &\triangleq x(t) + x_k \operatorname{sgn} \dot{x}(t) - x_0 \;=\; x(t) + x_k \operatorname{sgn} \dot{y}(t) - x_0 \\ &= x(t) + [\, \bar{x}_k + \Delta x_k \operatorname{sgn} \dot{y}(t) \,] \operatorname{sgn} \dot{y}(t) - x_0 \;=\; x(t) + \bar{x}_k \operatorname{sgn} \dot{y}(t) + \Delta x_k - x_0 \;, \end{aligned} \tag{15}$$

the solution of which now also contains the asymmetric rate of decay

$$\sigma \triangleq \bar{\sigma} + \Delta\sigma \operatorname{sgn} \dot{y}(t) \;, \tag{16}$$

where of course $\dot{y}(t) \equiv \dot{x}(t)$ as before, within regions of unidirectional motion.

SYSTEM PARAMETER IDENTIFICATION

The Logarithmic Decrement Method

The logarithmic decrement method is a popular way to identify an underdamped system by examining the envelope and frequency of its oscillation. It has its roots in work pioneered over a century ago; recently it was revisited by Feeny and Liang [Feeny and Liang (1996)] to include estimation of Coulomb friction in addition to the usual viscous vibration damping; the derivation is elucidated below.

Given an underdamped oscillation (7), its velocity may be written

$$\dot{y}(t) = Ae^{-\sigma t}\left[\omega_d \cos(\omega_d t - \phi_0) - \sigma \sin(\omega_d t - \phi_0)\right] \ , \tag{17}$$

which is nil ($\dot{y}(t) = 0$) when the displacement $y(t)$ is at a maximum or minimum (at a *peak*). Denoting the n^{th} peak displacement as $y_n(t)$, and the corresponding velocity $\dot{y}_n(t) = 0$, the angle of oscillation $\omega_d t - \phi_0$ at peak n can be written as

$$\tan(\omega_d t - \phi_0) = \frac{\omega_d}{\sigma} = \frac{\sqrt{1-\zeta^2}}{\zeta} \ . \tag{18}$$

The phase at the n^{th} peak is

$$\tan\phi_0 = -\frac{\omega_d y_n}{\dot{y}_n + \sigma y_n} \ , \tag{19}$$

which when substituted into the trigonometric expansion of (18) yields

$$\tan\omega_d t = \frac{\omega_d \dot{y}_n}{\sigma \dot{y}_n + \omega_0^2 y_n} \ . \tag{20}$$

This expression is always nil because $\dot{y}_n = 0$ by definition, and thus $\omega_d t = n\pi$, from which it follows that

$$\sigma t = \zeta\omega_0 t = n\pi \frac{\zeta}{\sqrt{1-\zeta^2}} = n\pi \frac{\sigma}{\omega_d} \tag{21}$$

and

$$\sin\omega_d t = \sin n\pi = 0 \qquad \text{and} \qquad \cos\omega_d t = \cos n\pi = (-1)^n \tag{22}$$

at the n^{th} oscillation peak.

Substituting $\dot{y}_n = 0$ into equations (11) and (19) further yield, respectively,

$$A = \frac{\omega_0}{\omega_d}\, y_0 \,, \quad \sin\phi_0 = -\sqrt{1-\zeta^2}\,, \quad \text{and} \quad \cos\phi_0 = \zeta \ . \tag{23}$$

Substituting the results of equations (21) - (23) into the trigonometric expansion of (7) finally gives the n^{th} peak displacement

$$y_n = (-1)^n y_0 e^{-n\pi\beta} = -y_{n-1}e^{-\pi\beta} \qquad \forall \quad n > 0 \ , \tag{24}$$

where the *logarithmic decrement* is defined as the logarithm of the ratio between successive peaks,

$$\pi\beta \triangleq -\ln\left\{-\frac{y_n}{y_{n-1}}\right\} = -\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\frac{\pi\sigma}{\omega_d} \ , \tag{25}$$

which, incidentally, is the equivalent of $(-\pi\cot\phi_0)$ at the oscillation peaks.

Asymmetric Friction Estimation. The traditional logarithmic decrement method may be extended to additionally estimate frictional asymmetry. Asymmetric viscous friction can be expressed in terms of two values β^+ and β^- such that equation (24) becomes

$$y_n = -y_{n-1}e^{-\pi\beta} \ , \tag{26}$$

where either $\beta = \beta^+$ or $\beta = \beta^-$ depending on whether n is even or odd (exactly which is unimportant, as long as the user is consistent with the adopted notation). For symmetric $\beta = \beta^+ = \beta^-$ this collapses once again into equation (24).

When the asymmetric dry and viscous friction of equations (15) and (16) are substituted into the oscillation peaks given by (26), the frictional asymmetry of the oscillation becomes visible:

$$x_n + x_{n-1}e^{-\pi\beta^+} = -\left(1 + e^{-\pi\beta^+}\right)\left[\Delta x_k + (-1)^n \bar{x}_k\right] \tag{27a}$$

$$x_{n-1} + x_{n-2}e^{-\pi\beta^-} = -\left(1 + e^{-\pi\beta^-}\right)\left[\Delta x_k + (-1)^{n-1} \bar{x}_k\right] \ , \tag{27b}$$

with the substitution $\operatorname{sgn}\dot{y} = \operatorname{sgn}\dot{x} = (-1)^n$, where n is *odd* if motion starts in the negative direction (either $y_0 > 0$ and/or $\dot{y}_0 < 0$), and *even* if motion starts in the positive direction (either $y_0 < 0$ and/or $\dot{y}_0 > 0$). The (+) and (−) superscripts on β denote the direction of motion ($\operatorname{sgn}\dot{y}$) between oscillation peaks, and the associated positive or negative viscous friction bias introduced by the asymmetric damping of equation (16).

It is important to note that the $\operatorname{sgn}\dot{y}$ term in equations (15) and (16) is constant for all motion between points n and $n-1$; in other words, the signum function changes sign only *in-between* the oscillation peaks, not *at* the peaks themselves, where instead it equals zero. Successive peak displacements are used to infer information about the motion between those peaks, not at the peaks themselves, hence the correct sign of the signum function as applied to the frictional term $\bar{x}_k$ at those peaks should be that of the velocity between the same. This explains why the $(-1)^n\bar{x}_k$ term can be gathered with the Δx_k term on the right hand sides of equation (27).

Notice furthermore that the expression solved for $x(t)$ in equation (15) is equivalent to the solution for $y(t)$ as defined, but for the sign reversal of the kinetic friction terms. Because of this similarity in the solution form, y_n may be substituted for x_n in (27) by reversing the sign on the right-hand sides of the equations. In this manner the friction can be estimated from the actual data points $y(t)$ (since the friction-free displacements $x(t)$ are unknown before the friction is estimated).

Solving for the decrement ratios,

$$e^{-\pi\beta^+} = -\frac{y_n - y_{n-2}}{y_{n-1} - y_{n-3}} \qquad \text{and} \qquad e^{-\pi\beta^-} = -\frac{y_{n-1} - y_{n-3}}{y_{n-2} - y_{n-4}} \ . \tag{28}$$

These results are an extension of equation (25) which take advantage of the observation that $y_n - y_{n-2} = x_n - x_{n-2}$ to remove the kinetic friction contribution from the viscous friction estimation.

Solving (27) for the kinetic friction parameters, the kinetic friction is given by

$$\bar{x}_k = \frac{1}{2(-1)^n}\left(\frac{y_n + y_{n-1}e^{-\pi\beta^+}}{1 + e^{-\pi\beta^+}} - \frac{y_{n-1} + y_{n-2}e^{-\pi\beta^-}}{1 + e^{-\pi\beta^-}}\right) \tag{29a}$$

and

$$\Delta x_k = \frac{1}{2}\left(\frac{y_n + y_{n-1}e^{-\pi\beta^+}}{1 + e^{-\pi\beta^+}} + \frac{y_{n-1} + y_{n-2}e^{-\pi\beta^-}}{1 + e^{-\pi\beta^-}}\right) + x_0 \ . \tag{29b}$$

The asymmetric kinetic friction may be determined directly from the logarithmic decrement and oscillation peaks as per equations (29). Using the logarithmic decrement values in equations (28), the respective viscous damping ratios may also be determined from equation (25):

$$\zeta^{\pm} = \sqrt{\frac{(\beta^{\pm})^2}{\pi^2 + (\beta^{\pm})^2}} \ , \tag{30}$$

from which

$$\bar{\zeta} = \frac{\zeta^+ + \zeta^-}{2} \qquad \text{and} \qquad \Delta\zeta = \left|\frac{\zeta^+ - \zeta^-}{2}\right| \ . \tag{31}$$

Identification Procedure. When the viscous friction is asymmetric ($\Delta\zeta \neq 0$), the oscillation will also be asymmetric, complicating the natural frequency estimation. Whereas the natural frequency of the system is of course constant, the damped natural frequency, as evident in the asymmetric response, takes one of two values depending on the direction of motion:

$$\omega_d^{\pm} = \frac{\pi}{\Delta t^{\pm}} \ , \tag{32}$$

where the Δt denotes the time difference between successive oscillation peaks:

$$\Delta t^{+} \triangleq t_n - t_{n-1} \qquad \text{and} \qquad \Delta t^{-} \triangleq t_{n-1} - t_{n-2} \; . \tag{33}$$

The natural frequency of the system is then given by

$$\omega_0 = \frac{\omega_d^{\pm}}{\sqrt{1-\zeta^{\pm}}} \; . \tag{34}$$

(Naturally, if these two equivalent terms are averaged, the estimation can also be improved against noise in the data.)

Note lastly that the equations presented in this section are first-order approximations, using the fewest number of data for each estimation. The approximations may be extended to higher dimensions by further combining the successive terms defined in equations (27).

DISCUSSION

Table 1 shows the results of estimation on simulated oscillations for various friction conditions. The results are generally accurate to within five per cent.

Generally speaking, the more oscillations, the better the log decrement estimation. On the other hand, the method works reasonably well even with the minimum of four oscillation peaks, depending on the signal-to-noise ratio of the oscillation data.

Since the estimation of parameters using the simulated response has an accuracy only on the order of about five per cent, this would be the anticipated baseline accuracy for analysis using experimental data. Unlike some other methods, the logarithmic decrement method uses only a fraction of the information avaliable in the data (the peak times and values). This has the advantage of allowing parameter estimation in the presence of noise, however this is traded off for a moderate loss in accuracy. It is nonetheless also one of the few methods available for determining the friction asymmetry in such a straightforward manner.

When the estimation for k is corrected with the information that there should be no spring force in the system, the corrected mass estimate is within one per cent of the true value. This shows that quality of the mass estimation depends directly on the quality of the frequency and damping data measured during oscillation. In practice, the frequency estimation is quite accurately accomplished by examining the zero- (or mean-) crossings of the oscillation; it is the damping estimates determined by the log decrement method which generally exhibit the larger of the estimation errors.

CONCLUSION

We show how to determine the asymmetric kinetic and viscous friction via an extended treatment of the logarithmic decrement method. For overdamped systems, the same identification may be performed by applying the new parametric oscillation method. The latter also provides a mechanism for estimating the system mass, and hence its mass-based parameters.

We would like to invite the reader to download and examine the C, Matlab and data files used to prepare the simulations and experimental analysis, by visiting our web address. We would furthermore appreciate any comments or feedback via e-mail.

ACKNOWLEDGMENTS

We wish to thank the National Science Foundation for their sustained support of our research in mechatronics for machine tools, under NSF grant No. GER-9354913 [Craig et al. (1993)]. We would also like to thank the following individuals for their invaluable contributions and assistance during our work: Prof. C. Jim Li, Ph.D.; Jeongmin Lee, M.S.; and Laurent Chazeau, Ph.D.

*

REFERENCES

Coulomb, C. A., 1785, "Théorie des Machines Simples, En Ayant Égard au Frottement de Leur Parties, et à la Roideur du Cordages," *Mém. des Math. Phys.*, Vol. 10, pp. 161–342.

Craig, K. C. et al., 1993, "NSF Graduate Research Traineeships for Mechatronics in Machine Tool Research," Rensselaer Dept. of ME, AE & M proposal to the NSF, originating NSF funding for the research presented in this article.

Feeny, B. F. and Liang, J. W., 1996, "A Decrement Method for the Simultaneous Estimation of Coulomb and Viscous Friction," *J. Sound and Vib.*, Vol. 195, pp. 149–154.

PMI Motion Technologies, 1986a, *U12M4T ServoDisc*, Kollmorgen Corporation, 49 Mall Drive, Commack, NY 11725-5703, U-Series ServoDisc specifications edn., Reference.

PMI Motion Technologies, 1986b, *U16M4T ServoDisc*, Kollmorgen Corporation, 49 Mall Drive, Commack, NY 11725-5703, U-Series ServoDisc specifications edn., Reference.

Prakah-Asante, K. O. et al., 1993, "Design, Construction and Testing of a Singe-Axis Servomechanism for Control Experiments Involving Coulomb Friction, Backlash and Joint Compliance," *J. Eng. Design*, Vol. 4, No. 4, pp. 305–329.

Tüfekçi, C. S. et al., 1998, "Mechatronic Design of an Inverted Pendulum System for Engineering Education," in: *Proceedings of the 6th UK Mechatronics Forum International Conference (Mechatronics '98)*, eds. J. Adolfsson and J. Karlsén, Mechatronics '98, UK Mechatronics Forum Centre for Intelligent Automation, North Holland-Elsevier Science Publishers, Amsterdam / New York City / Oxford, pp. 685–691.

Walczyk, D. F., 1991, "Design and Implementation of an Experimental Computer-Controlled Single-Axis Manipulator," M.S. Thesis, Rensselaer Polytechnic Institute, Troy, New York, Kevin C. Craig, advisor.

APPENDIX: PARAMETRIC OSCILLATION DATA

TABLE 1: Simulation: Log-Decrement Identification.

	ω_0 (rad/s)	$\bar{f}_k \pm \Delta f_k$ (Nm)	$\bar{c} \pm \Delta c$ (Nms/rad)
simulated	3.162	0.2000 ± 0.5000	0.4000 ± 0.3000
estimated	3.277	0.2148 ± 0.5370	0.4140 ± 0.3109
% error	3.6%	$7.4\% \pm 7.4\%$	$3.5\% \pm 3.6\%$
simulated	4.000	0.2000 ± 0.5000	0.3000 ± 0.4000
estimated	4.129	0.2130 ± 0.5326	0.3095 ± 0.4129
% error	3.2%	$6.5\% \pm 6.5\%$	$3.2\% \pm 3.2\%$
simulated	4.000	0.0700 ± 0.1300	0.3000 ± 0.0000
estimated	4.023	0.0715 ± 0.1352	0.3015 ± 0.0002
% error	0.6%	$2.1\% \pm 4.0\%$	$0.5\% \pm 0.2\%$
simulated	4.000	2.0000 ± 2.0000	0.6250 ± 0.1250
estimated	4.024	2.0098 ± 2.0316	0.6155 ± 0.1134
% error	0.6%	$0.5\% \pm 1.6\%$	$1.5\% \pm 9.3\%$
simulated	2.000	0.0000 ± 0.0000	0.4000 ± 0.3000
estimated	2.094	0.0000 ± 0.0000	0.4184 ± 0.3141
% error	4.7%	$0.0\% \pm 0.0\%$	$4.6\% \pm 4.7\%$
simulated	2.000	0.2000 ± 0.0000	0.5000 ± 0.0000
estimated	2.015	0.2030 ± 0.0001	0.5034 ± 0.0000
% error	0.8%	$1.5\% \pm 0.1\%$	$0.7\% \pm 0.0\%$
simulated	4.000	6.2500 ± 5.7500	0.6250 ± 0.1250
estimated	4.026	6.3193 ± 5.7178	0.6293 ± 0.1288
% error	0.7%	$1.1\% \pm 0.6\%$	$0.7\% \pm 3.0\%$
simulated	4.000	0.4000 ± 0.3000	0.1000 ± 0.0000
estimated	4.114	0.4213 ± 0.2836	0.1013 ± 0.0008
% error	2.9%	$5.3\% \pm 5.5\%$	$1.3\% \pm 0.8\%$

TABLE 2: Pendulum System Identification.

(Published values from PMI Motion Technologies (1986a) and Tüfekçi et al. (1998).)

	motor only			motor + cart		
	published	measured	error	published	measured	error
Δt (ms)	10.84	9.2838	14%	–	10.548	–
m (kg)	1.4250	1.6466	16%	2.3900	2.4223	1.4%
k (N/m)	0 (nil)	2.8790	9.3%	0 (nil)	+8.9827	–
$\bar{c}$ (Ns/m)	0.3656	0.3727	1.9%	–	0.8043	–
Δc (Ns/m)	0 (nil)	-0.0096	–	–	+0.3268	–
$\bar{f}_k$ (N)	3.7330	-5.1676	38%	–	7.9340	–
Δf_k (N)	0 (nil)	1.5043	–	–	+0.7163	–

TABLE 3: Test Bed System Identification.

(Published values from PMI Motion Technologies (1986b), Walczyk (1991), and Prakah-Asante et al. (1993).)

	A only			A + B			A + B + C		
	published	measured	error	published	measured	error	published	measured	error
Δt (ms)	13.77	12.24	11%	–	10.76	–	–	10.63	–
J (Nms^2)	0.0115355	0.01194	3.5%	0.0150626	0.01744	16%	0.0339399	0.03136	7.6%
k (Nm)	0 (nil)	0.001034	1.6%	0 (nil)	0.1492	24%	nil (0)	-0.01168	1.9%
$\bar{c}$ (Nms)	0.0007457	0.002355	216%	0.0018657	0.001233	34%	0.0029857	0.004514	51%
Δc (Ns/m)	–	0.0002458	–	–	0.000848	–	–	0.000858	–
$\bar{f}_k$ (Nm)	0.109456	-0.07233	34%	0.1416169	-0.1921	36%	0.1711679	-0.1968	15%
Δf_k (Nm)	–	0.008622	–	–	0.0243	–	–	0.005588	–