# Identification of Arbitrarily Overdamped Second-Order Systems via Parametric Harmonic Oscillation

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## ABSTRACT

We introduce a novel feedback technique, called *parametric harmonic oscillation*, whereby even highly overdamped systems can be made to mimick underdamped free harmonic vibration. This allows one to extend identification methods intended for underdamped systems to any second-order system, including those which are heavily overdamped. We also show how to use the parametric harmonic oscillation method to reveal the physical mass, viscous friction, and stiffness parameters of the system, as well as the usual mass-dimensionalised natural frequency and damping coefficient. The applicability of the new technique is demonstrated in simulation.

## INTRODUCTION

### Simple Harmonic Motion

Consider the free (homogeneous) harmonic oscillation of the piecewise linear, time-invariant, second-order system

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + f_c = 0\ . \tag{1}$$

The dynamics of this system ($x(t)$ and its derivatives) describe a simple harmonic motion.

### Transient (Unforced) Harmonic Oscillation

The transient response is that observed when the system is unforced ($F(t) \equiv 0$). The general (homogeneous) solution may be expressed as

$$x(t) = Ae^{-\sigma t}\sinh(\omega_d t - \phi_0) \tag{2}$$

where the *rate of decay*

$$\sigma \triangleq \frac{c}{2m}\ , \tag{3}$$

which relates the *damped natural frequency*

$$\omega_d \triangleq \sqrt{\sigma^2 - \omega_0^2} \tag{4}$$

to the *(undamped) natural frequency*

$$\omega_0 \triangleq \sqrt{\frac{k}{m}} \; . \tag{5}$$

The oscillation has an *amplitude* (also called *envelope*)

$$A^2 = \frac{\dot{x}_0^2 + 2\sigma\dot{x}_0 x_0 + \omega_0^2 x_0^2}{\omega_d^2} \tag{6}$$

and *phase*

$$\tanh\phi_0 = -\frac{\omega_d x_0}{\dot{x}_0 + \sigma x_0} \; , \tag{7}$$

given the initial conditions on position and velocity

$$x_0 \triangleq x(t_0) \qquad \text{and} \qquad \dot{x}_0 \triangleq \dot{x}(t_0) \; . \tag{8}$$

These equations are valid for *all* real first- and second-order oscillations, regardless of damping or stiffness, provided that the physical quantities $m > 0$, $c \geq 0$ and $k \geq 0$.

**Overdamped Response.** The system is *overdamped* when $\sigma^2 > \omega_0^2$. In this case equation (2) may be applied without modification. The special case when $k = 0$ ($\omega_0 = 0$) results in an equivalent first-order system, which has no static restoring (spring) force. Usually this first-order response is analysed in terms of a first-order system, for example

$$m\dot{z}(t) + cz(t) = F(t) \; , \tag{9}$$

from which the response is recovered by integrating the solution $z(t) \triangleq \dot{x}(t)$. However, the very same response may be obtained simply by substituting $k = 0$ directly into equations (2) through (8).

**Critically-damped Response.** For systems with non-zero stiffness $k > 0$, the nondimensional *damping coefficient* is written as one of the standard equations

$$\zeta \triangleq \frac{\sigma}{\omega_0} = \frac{c}{c_c} = \frac{c}{2m\omega_0} = \frac{c}{2\sqrt{mk}} \; . \tag{10}$$

The meaning of the damping coefficient $\zeta$ as the ratio between the viscous damping coefficient $c$ to the *critical damping coefficient* $c_c$ is that the critical value $c = c_c$ demarcates the oscillatory and non-oscillatory system responses. For underdamped systems, $0 \leq \zeta < 1$.

When $\sigma^2 = \omega_0^2$ ($\zeta = 1$) the system is *critically damped*. In this case, both $\omega_d = 0$ and $\phi_0 = 0$. Observing that

$$\lim_{\omega_d \to 0} \frac{\sinh \omega_d t}{\omega_d} = \lim_{\omega_d \to 0} \frac{\sin \omega_d t}{\omega_d} = t \; , \tag{11}$$

substitution into equation (2) results in the familiar expression for the critically-damped response,

$$x(t) = te^{-\sigma t}\sqrt{\dot{x}_0^2 + 2\sigma\dot{x}_0 x_0 + \omega_0^2 x_0^2} \; . \tag{12}$$

Notice that since $\sigma = \omega_0$, either may be used interchangeably in the final expression.

**Underdamped Response.** The system is *underdamped* when $\sigma^2 < \omega_0^2$. In this case, $\omega_d$ (as defined) will be an imaginary number, so it is customarily redefined to be the complex conjugate of equation (4). Thus, using $i\omega_d$ and $i\phi_0$ instead of $\omega_d$ and $\phi_0$, respectively, in equations (6) and (7), and then substituting into (2),

$$x(t) = -iAe^{-\sigma t}\sinh(i\omega_d t - i\phi_0) \; . \tag{13}$$

This can be rewritten via the identity

$$-i\sinh(ix) = \sin(x) \ , \tag{14}$$

to reveal the familiar equation for underdamped oscillation,

$$x(t) = Ae^{-\sigma t}\sin(\omega_d t - \phi_0) \ . \tag{15}$$

The overdamped and underdamped equations are therefore mathematically equivalent; redefining the damped natural frequency to maintain its realness for underdamped oscillations is generally a matter of convenience for educational and interpretive purposes, since usually people are more intimately familiar with the behaviour of regular sines and cosines than of their hyperbolic equivalents.

## SYSTEM PARAMETER IDENTIFICATION

The ultimate objective of system identification is to discover the physical parameter values which ascribe the system's behaviour. For the systems described in this treatment, the values of mass $m$, viscous friction coefficient $c$, and spring constant $k$ should be estimated. It is difficult to measure precisely the values of these parameters when they differ greatly in relative scale. For example, when there is very little damping ($\zeta \approx 0$), the behaviour is predominantly oscillatory and $c$ will be difficult to estimate with good confidence. Similarly, $k$ is nearly impossible to estimate when the natural frequency of the system is close to naught ($\omega_0 \approx 0$). The system mass $m$ is also hard to determine when it is small relative to $c$ and $k$. Ideally, therefore, each of these parameters would be close to one another in relative scale. However, this is impossible to achieve physically if the system parameters are for some reason unchangeable. The problem is that oscillatory motion is required to implement most available time-domain identification techniques with reasonable precision.

### Forced Harmonic Oscillation

Oscillatory motion may be induced in the system by applying an external harmonic force $F(t)$. After the transient response of the system has subsided ($Ae^{-\sigma t}$ in equation (2) is acceptably small for some $t \gg t_0$), the *steady-state* response $x(t \gg t_0)$ will be an oscillation at the same frequency as that of the input $F(t)$.

For example, let the *forcing function* be defined as

$$F(t) \triangleq B\sin(\omega t - \theta) \ . \tag{16}$$

The persistent system response to this force is then

$$x_p(t) \triangleq x(t \gg t_0) = C\sin(\omega t - \phi) \ . \tag{17}$$

Substituting this $F(t)$, with the associated $x_p(t)$ and its derivatives, into (1),

$$(k - m\omega^2)\sin(\omega t - \phi) + c\omega\cos(\omega t - \phi) = \frac{B}{C}\sin(\omega t - \theta) \ . \tag{18}$$

Expanding the trigonometric functions and equating the resulting terms,

$$(k - m\omega^2)\cos\phi + c\omega\sin\phi = \frac{B}{C}\cos\theta \tag{19a}$$

and

$$-(k - m\omega^2)\sin\phi + c\omega\cos\phi = -\frac{B}{C}\sin\theta \ . \tag{19b}$$

Alternately (post-)multiplying by $\sin\phi$ and $\cos\phi$, these equations may be combined to produce the compact result

$$(k - m\omega^2) = R\cos\rho \qquad \text{and} \qquad c\omega = R\sin\rho \ , \tag{20}$$

where the *relative stiffness* and *relative phase (lag)* are

$$R \triangleq \frac{B}{C} \qquad \text{and} \qquad \rho \triangleq \phi - \theta \ . \tag{21}$$

(Note that $C$ is an inherent function of $\omega$, with units of mass, whereas $B$ is a constant with units of force.)

Applying successive forcing frequencies $\omega_i$, the mass may be estimated as

$$m = \frac{R_j \cos\rho_j - R_i \cos\rho_i}{\omega_i^2 - \omega_j^2} , \tag{22a}$$

the damping as

$$c = \frac{\omega_i R_j \sin\rho_j + \omega_i R_j \sin\rho_j}{2\omega_i \omega_j} \tag{22b}$$

(which is merely the average of successive measurements) and the stiffness as

$$k = \frac{\omega_i^2 R_j \cos\rho_j - \omega_j^2 R_i \cos\rho_i}{\omega_i^2 - \omega_j^2} . \tag{22c}$$

## Parametric Harmonic Oscillationusing Proportional and Derivative Feedback

An alternative to forced harmonic oscillation is the parametric harmonic oscillation method, which resolves the need for oscillation by artificially modulating the energy dissipation and storage of the system. Rather than forcing the system with a sinusoid, the damping and stiffness of the system are artifically manipulated, producing various (artificial) free harmonic oscillations. Consider the forced second-order system

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) = -\left[\, D_i \dot{x}(t') + P_i x(t') \,\right] , \tag{23}$$

where $D$ is a derivative feedback constant with the same units as $c$, and $P$ is a proportional feedback with the same units as $k$. The control force is presumed to be delayed by some time interval $\Delta t \triangleq t - t'$, due to any of a number of realistic factors, such as measurement and computation delays, or the mechanical or electrical time constant of the motor and amplifier providing the parametric feedback.

Because the whole idea behind parametric harmonic oscillation is to produce an artificially underdamped system, we can assume that the displacements have the form described by equation (15). Therefore

$$\begin{aligned} x(t') &= Ae^{-\sigma t'} \sin\left(\omega_d t' - \phi_0\right) = Ae^{-\sigma(t-\Delta t)} \sin\left[\omega_d (t - \Delta t) - \phi_0\right] \\ &= Ae^{-\sigma t} e^{\sigma \Delta t} \left[\cos(\Delta\phi)\sin(\omega_d t - \phi_0) - \sin(\Delta\phi)\cos(\omega_d t - \phi_0)\right] , \end{aligned} \tag{24}$$

and its derivative

$$\dot{x}(t') = Ae^{-\sigma t} e^{\sigma\Delta t} \left\{ \left[\omega_d \sin(\Delta\phi) - \sigma\cos(\Delta\phi)\right] \sin(\omega_d t - \phi_0) + \left[\sigma \sin(\Delta\phi) + \omega_d \cos(\Delta\phi)\right] \cos(\omega_d t - \phi_0) \right\} , \tag{25}$$

where the phase lag caused by time delay $\Delta t$

$$\Delta\phi \triangleq \omega_d \Delta t . \tag{26}$$

We would like to rewrite $x(t')$ and its derivative wholly in terms of $x(t)$ and its derivatives, since only these states are measurable. From equation (15) and its derivative, we know that

$$Ae^{-\sigma t} \cos(\omega_d t - \phi_0) = \frac{\dot{x} + \sigma x}{\omega_d} . \tag{27}$$

Substituting this result, along with equation (15), into equations (24) and (25), respectively, we obtain the simplified result

$$x(t') = \left[(\alpha_1 - \sigma\alpha_2)\, x(t) - \alpha_2 \dot{x}(t)\right] . \tag{28a}$$

Noting that the system is underdamped by virtue of the parametric feedback, the complex conjugate of equation (4) can be used to make the substitution $\omega_0^2 = \omega_d^2 + \sigma^2$, from which the derivative of $x(t)$ is

$$\dot{x}(t') = \left[\omega_0^2 \alpha_2 x(t) + (\alpha_1 + \sigma\alpha_2)\, \dot{x}(t)\right] , \tag{28b}$$

where

$$\alpha_1 \triangleq e^{\sigma\Delta t}\left[\cos(\Delta\phi) - \frac{\sigma}{\omega_d}\sin(\Delta\phi)\right] \qquad \text{and} \qquad \alpha_2 \triangleq e^{\sigma\Delta t}\frac{1}{\omega_d}\sin(\Delta\phi)\ . \tag{29}$$

Now, gathering the terms of equation (23) yields the homogenous form

$$m\ddot{x}(t) + \left[c\dot{x}(t) + D_i\dot{x}(t')\right] + \left[kx(t) + P_i x(t')\right] = 0\ . \tag{30}$$

Substituting equations (28a) and (28b) and gathering like terms produces the simplified homogenous equivalent

$$m\ddot{x}(t) + c_i\dot{x}(t) + k_i x(t) = 0\ , \tag{31}$$

where

$$c_i \triangleq c + c_i' = c + (\alpha_{1i} + \sigma\alpha_{2i})\, D_i - \alpha_{2i}P_i \tag{32a}$$

and

$$k_i \triangleq k + k_i' = k + (\alpha_{1i} - \sigma\alpha_{2i})\, P_i + \omega_0^2\alpha_{2i}D_i\ , \tag{32b}$$

with a solution of the same form as equation (15). Here the subscript $i$ denotes the analysis of the $i^{th}$ response oscillation of the system, obtained using predetermined proportional and derivative feedback gains $P_i$ and $D_i$. The system's harmonic oscillation may thus be controlled as if the feedback parameters were built into the physics of the system itself. Notice that when $c_i' > 0$, damping is added to the system, and when $c_i' < 0$, damping is removed from the system. Similarly, when $k_i' > 0$, the system is stiffened, and when $k_i' < 0$, the system's stiffness is relaxed.

Since the feedback gains are predetermined, the rate of decay $\sigma$ and natural frequency $\omega_0$ may be estimated in spite of the feedback time delay $\Delta t$. In the absence of a time delay ($\Delta t = 0$), $\alpha_1 = 1$ and $\alpha_2 = 0$, so that $c_i' \equiv D_i$ and $k_i' \equiv P_i$. In the absence of parametric feedback, of course ($c_i' \equiv k_i' \equiv 0$), equations (32a) and (32b) reduce to $c_i \equiv c$ and $k_i \equiv k$, respectively, as would be expected.

**(Pseudo-)Free Parametric Harmonic Oscillation.** The significance of the parametric harmonic oscillation technique is that, for example, heavily overdamped systems ($c \gg k$) can be identified using methods requiring oscillations, and heavily underdamped systems ($c \ll k$) can be identified more quickly by introducing appropriate damping. The harmonic oscillation is now controlled by appropriate selection of values for $P_i$ and $D_i$.

Using the parametric values $c_i$ and $k_i$, equations (3) and (5) yield

$$c_i = 2m\sigma_i = 2m\zeta_i\omega_{0i} \qquad \text{and} \qquad k_i = m\omega_{0i}^2\ . \tag{33}$$

Equations (32a) and (32b) yield

$$c_i' = c_i - c \qquad \text{and} \qquad k_i' = k_i - k\ , \tag{34}$$

and substituting these into equations (33),

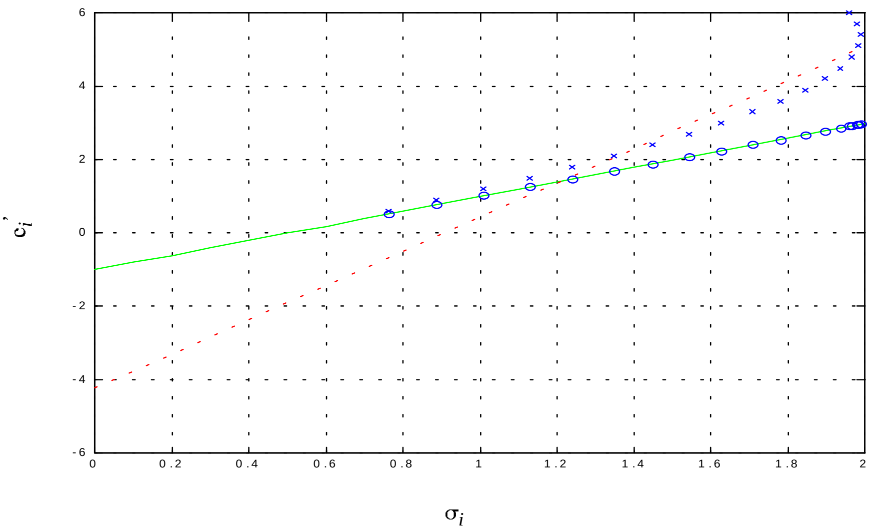
$$c_i' = 2m\sigma_i - c \qquad \text{and} \qquad k_i' = m\omega_{0i}^2 - k\ . \tag{35}$$

Both these equations have the form of straight lines. Taking the multiplier of $m$ as the "independent" variable ($2\sigma_i$ and $\omega_{0i}^2$, respectively), and the primed, left-hand side as the "dependent" variable ($c_i'$ and $k_i'$, respectively), the slope of either equation will be $m$, whereas the negative of the intercept of the former equation will yield $c$, and that of the latter, $k$. Thus the physical parameters can be determined directly, using a linear least-squares fit, provided two or more oscillation measurements $i$ are made.

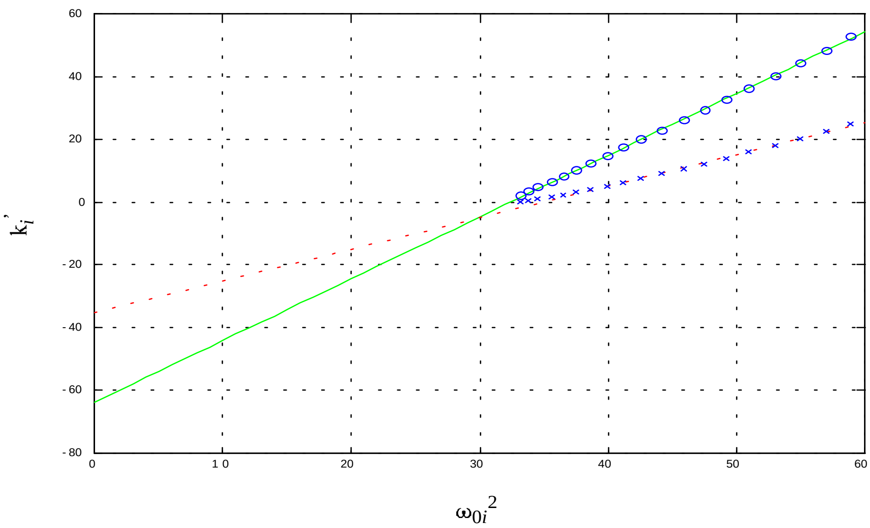
## SIMULATION RESULTS

Table 1 shows the results of a parametric harmonic oscillation for a system with the physical values shown. The results are obtained from the logarithmic decrement analysis of 19 oscillations, with $P_i$ varying from 0.25 N/m to 25 N/m in steps of 1.375, and $D_i$ varying from 0.6 Ns/m to 6 Ns/m in steps of 0.3, respectively.

Figures 1(a) and 1(b) show the least-squares fits to the $c_i'$ and $k_i'$ parameters. The lines denote the linear fits, the circles denote the values adjusted for a time delay $\Delta t = 0.1$ s, and the crosses denote the values without time delay compensation. The dashed lines indicate the least-squares curve fit without time delay compensation. Clearly, without compensation, the friction estimations do not fit a line, and the stiffness estimations have the wrong slope even though they do fit a straight line. In both figures, the slopes of the compensated fits can be seen to equal the correct mass, and their intercepts equal the correct viscous friction, and stiffness, respectively.




(a) Actual vs. Estimated Mass (slope) and Viscous Friction (intercept).




(b) Actual vs. Estimated Mass (slope) and Stiffness (intercept).

## DISCUSSION

Intuitively, the more oscillations $i$ that are available, the more confidence can be placed in estimations made in this manner. Take note, however, that when dealing with real data containing noise, the intercept estimations will suffer regardless of the number of oscillations if the respective slope is small. Thus, the estimate for $c$ may be poor if there is little change in the damping between successive oscillations, and that of $k$ may suffer if there is little change in the frequency between successive oscillations. This indicates that in spite of the flexibility offered by the parametric harmonic oscillation approach, care must be taken to sweep across some appreciable range of both damping and frequency. Fortunately, this can be accomplished by proper selection of the feedback parameters $P_i$ and $D_i$, which can be varied as necessary.

Counteracting the usefulness of many oscillations is the fact that when using discrete-time measurements of the

**TABLE 1: Parametric Oscillation Results with and without Time Delay Compensation.**

| | actual value | with compensation | without compensation |
|---|---|---|---|
| $m$ (kg) | 2 | 1.9884 | 1.0169 |
| $c$ (Ns/m) | 1 | 1.0035 | 4.2321 |
| $k$ (N/m) | 64 | 63.7668 | 35.1812 |

response, higher frequencies, which result in more oscillation, will reduce the effective resolution of the frequency estimation, because each oscillation cycle will occur over a smaller range of measurements for a fixed sampling time. This should be taken into consideration when applying the parametric harmonic feedback using a digital control system.

The viscous friction estimation suffers in the experimental situation when there is a time delay in the PD-feedback loop used for parametric harmonic oscillation. The severity of this problem depends on the length of the delay relative to the period of each harmonic oscillation. Because larger system masses produce slower oscillations, it is typically the systems with smaller mass which exhibit this adverse effect. It may therefore be advisable to add mass to the system for the experimentation, and then subtract this known value from the final estimation, in order to circumvent the time delay effect when it cannot be estimated directly. Mass estimation is relatively insensitive to the accuracy of the friction estimation using this method. However, accurate estimation of the viscous friction and stiffness requires either a small relative time delay $\Delta t$, or that it is measured directly and compensated for as discussed. One way of doing this is by comparing the feedback command signal (for instance, the computer command to the motor amplifier) against the actual signal exciting the system (the actual motor current).

Derivative feedback requires velocity estimation, which for real-time digital control systems can be a challenge to implement with accuracy. Notice, however, that when there is an inherent time delay, the method can be applied using only proportional feedback, since although in such a situation $D_i \equiv 0$, both values $\alpha$ are non-zero, allowing $P_i$ to determine both $c_i'$ and $k_i'$. When the delay is small, however, $P_i$ must be changed dramatically to reduce noise in the estimation of $c$.

Lastly, it is worth noting that an interesting phenomenon occurs when there is appreciable kinetic friction but little viscous friction. When the maximum oscillation velocity is such that the maximum viscous friction is always less than the kinetic friction, it is possible to oscillate the system with negative damping and still achieve a stable response. In such a situation, the parametric damping coefficient $c_i' < 0$, and the envelope of the oscillation will have an accelerated slope, rather than a decelerated slope as for an exponential decay (the envelope will be bullet-shaped). For systems with a high kinetic friction this means that negative damping can be used to obtain the system parameters. Normally, in the absence of kinetic friction, negative damping causes exponential instability, so in any case negative damping should be used with caution. However, if the kinetic friction is high then negative damping will be the only way to obtain sufficient oscillation for successful identification using the parametric harmonic oscillation method, unless the PD feedback is augmented with a DC offset in the (alternating) direction of oscillation.

## CONCLUSION

The same identification techniques traditionally reserved for the free harmonic response of underdamped systems may now be performed on overdamped systems, by applying the new parametric oscillation method. This method also provides a mechanism for estimating the system mass, and hence its mass-based parameters.

Unlike the popular forced harmonic oscillation method, the parametric harmonic oscillation method has an intuitive physical association with the system damping and frequency, allowing conservative choices for the proportional (P) and derivative (D) feedback parameters which can be chosen to deliberately circumvent the system resonance frequencies. The PHO method is effectively the dual of the forced harmonic oscillation (sine-sweep) method, in the sense that both methods require only two measurements (varying PD gains for the former, and varying forcing frequency $\omega$ for the latter), and work on (piecewise) linear, second-order systems—basically mass-spring-damper systems with Coulomb friction. The main advantage is that now techniques for estimating system parameters based on the free (unforced) harmonic oscillation may also be applied to situations where it would otherwise be impossible, namely, insufficiently underdamped systems.

In summary, the parametric harmonic oscillation method is shown to provide a succinct way to estimate the physical system parameters of a system when a method for estimating $\zeta$ and $\omega_0$ is available.

We would like to invite the reader to download and examine the C, Matlab and data files used to prepare the simulations and experimental analysis, by visiting our web address. We would furthermore appreciate any comments or feedback via e-mail.

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